

On identifying the sound sources in a turbulent flow

M. E. Goldstein

National Aeronautics and Space Administration, Glenn Research Center, Cleveland, Ohio 44135
marvin.e.goldstein@nasa.gov

ABSTRACT

A space-time filtering approach is used to divide an unbounded turbulent flow into its radiating and non-radiating components. The result is then used to clarify a number of issues including the possibility of identifying the sources of the sound in such flows. It is also used to investigate the efficacy of some of the more recent computational approaches.

1. INTRODUCTION

It is impossible to identify the “sources” of sound without first defining what is actually meant by “sound”. Unfortunately, current understanding of unsteady compressible flows, especially turbulent shear flows, is still too rudimentary to give a completely general definition of this quantity. Most of the relevant theory is extremely quantitative and seems to fall back on Kovasznay’s (1953) decomposition of the small amplitude inviscid motion on a uniform flow into acoustic and non-acoustic vortical and entropic components, with the acoustic component being associated with the pressure fluctuations and the vortical/entropic component being independent of the pressure (see Smits and Dussauge, 1996). While it is widely recognized that this decomposition is invalid for real turbulent flows—where the pressure and vortical fluctuations can be strongly coupled— it is common practice to refer to the acoustic component of the motion even though it can’t be rigorously defined. Some progress along these lines was made by Lighthill (1952) who was able to implicitly identify such a component for a limited class of flows—namely those that are sufficiently localized in space. The vorticity is confined to a localized region in such flows and the external motion (especially at large distances) is completely characterized by the pressure fluctuations, which propagate (i.e. radiate) away from the flow.

Lighthill, of course, went on to show that these fluctuations are described by an inhomogeneous wave equation with a quadrupole-type source term. So in a certain sense the sound is actually generated by this source. But that source also generates all sorts of other (much more energetic but non-propagating) motions and it is impossible to distinguish what part of the source is actually responsible for generating the sound. So from a practical point of view, referring to this term as “the source of sound” is almost meaningless—especially since it is almost impossible to calculate or measure it with any great accuracy. What is really needed here is a way of identifying a source term that generates only the acoustic (i.e. the radiating) component of the motion. This would, if it were actually possible, provide a much more meaningful characterization of the elusive “acoustic source”. (An early attempt at this is outlined in Fedorchenko, 2001.)

The first step in finding such a source would be to provide a quantitative characterization of the acoustic (i.e. propagating component) of the motion--the seeds of which are actually contained in Lighthill's (1952) equation. It is possible to use that equation (Goldstein,2005) to demonstrate that only those components of the motion with space-time Fourier transforms that lie on the spherical surface in wave number or k - space with radius equal to the frequency, say ω , divided by the speed of sound at infinity, say c_0 , contribute to the propagating component of the motion (see figure 1).

The second step would be to use this characterization to divide the motion into its radiating and non-radiating components and to derive an equation for each of these components. The final step would be to identify a "source term" in the radiating component equation that only depends on the non-radiating component of the motion--*which would then be the desired sound source*. We would, of course, want the non-radiating component of the motion to have at least some chance of being a physically realizable flow, i.e. we would want it to satisfy the usual conservation laws of mass momentum and energy. It would also be nice if the radiating component of the motion was to satisfy linear equations—but this may be too much to ask.

2. THE BASIC EQUATIONS

It, therefore, seems appropriate to begin by dividing the dependent variables in the Navier Stokes equations, i.e. the density ρ , the pressure p , the velocity v_i and the enthalpy h into, say, a base flow component, which I denote here by over-bars and tildes and residual components, which are essentially defined by the equations

$$\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p', \quad h = \tilde{h} + h' \quad v_i = \tilde{v}_i + v'_i, \quad [2.1]$$

and to require that the base flow components satisfy the usual hyperbolic conservation laws, which can be written fairly compactly as (Goldstein and Leib,2007)

$$D_o \bar{\rho} = 0, \quad [2.2]$$

$$D_o \bar{\rho} \tilde{v}_i + \frac{\partial \bar{p}_e}{\partial x_i} = \frac{\partial}{\partial x_j} \tilde{e}_{ij} \quad [2.3]$$

$$D_o \left(\frac{\gamma \bar{p}_e}{\gamma - 1} + \frac{\bar{\rho}}{2} \tilde{v}^2 \right) - \frac{\partial \bar{p}_e}{\partial t} = \frac{\partial}{\partial x_i} \left(\frac{\tilde{e}_{4j}}{\gamma - 1} + \tilde{e}_{ij} \tilde{v}_j \right) \quad [2.4]$$

where the Latin indices range from 1 to 3, the summation convention is being used,

$$D_o f \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_j} (\tilde{v}_j f) \quad [2.5]$$

for any function f , we assume that the base flow variables, as well as the original Navier Stokes variables satisfy an ideal gas law equation of state, with γ being the specific heat ratio, \bar{p}_e denotes a pressure-like variable that can differ from the thermodynamic pressure, and $\tilde{e}_{\lambda j}$ with $\lambda = 1, 2, 3, 4$ denotes an, as yet, arbitrary 4x3 dimensional stress tensor. (Greek indices will always range from 1 to 4.) These equations include, among other things, the Euler equations, the Navier Stokes equations themselves, and most importantly for our purposes, the Favre filtered Navier Stokes equations.

We have previously shown (Goldstein, 2000, 2002, 2003, see also Goldstein and Leib, 2007) that the remaining residual variables are determined by the five formally linear equations

$$D_o \rho' + \frac{\partial}{\partial x_j} u_j = 0, \quad [2.6]$$

$$D_o u_i + u_j \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial}{\partial x_i} p'_e - \frac{\rho'}{\bar{\rho}} \frac{\partial}{\partial x_j} \tilde{\theta}_{ij} = \frac{\partial}{\partial x_j} e''_{ij} \quad [2.7]$$

and

$$D_o p'_e + \frac{\partial}{\partial x_j} \tilde{c}^2 u_j + (\gamma - 1) \left(p'_e \frac{\partial \tilde{v}_j}{\partial x_j} - \frac{u_i}{\bar{\rho}} \frac{\partial \tilde{\theta}_{ij}}{\partial x_j} \right) = \frac{\partial}{\partial x_j} e''_{4j} + (\gamma - 1) e''_{ij} \frac{\partial \tilde{v}_i}{\partial x_j}, \quad [2.8]$$

where

$$\tilde{\theta}_{ij} \equiv \delta_{ij} \bar{p}_e - \overline{e'_{ij}}, \quad [2.9]$$

is the total base flow stress tensor,

$$\tilde{c}^2 \equiv \gamma \bar{p} / \bar{\rho} \quad [2.10]$$

the square of the base flow sound speed. The true non-linearity of these equations is hidden in the non-linear dependent variables

$$p'_e \equiv p - \bar{p}_e + \frac{\gamma - 1}{2} \rho v'^2 = \left(p' + \frac{\gamma - 1}{2} \rho v'^2 \right) + \bar{p} - \bar{p}_e \quad [2.11]$$

$$u_i \equiv \rho v'_i, \quad [2.12]$$

as well as in the quadratically non-linear source strengths

$$e''_{\lambda i} \equiv e'_{\lambda i} - \tilde{e}_{\lambda i} \quad \lambda = 1, 2, 3, 4 \quad [2.13]$$

where $e'_{\lambda i}$ denotes the generalized Reynolds stress

$$e'_{\lambda i} \equiv -\rho v'_\lambda v'_i + \frac{\gamma-1}{2} \delta_{\lambda i} \rho v'^2 + \sigma_{\lambda i} \quad [2.14]$$

where

$$v'_4 \equiv (\gamma-1)h'_o \equiv (\gamma-1)\left(h' + \frac{1}{2}v'^2\right), \quad \sigma_{4i} \equiv -(\gamma-1)(q_i - \sigma_{ij}v'_j) \quad [2.15]$$

The former non-linearity causes no particular difficulty here because the variable of principle interest, namely the pressure-like variable p'_e , reduces to the ordinary pressure fluctuation p' in the far field where the sound is to be calculated. The latter non-linearity is discussed in section 4.

3. THE FILTERED EQUATIONS

We now show that the base flow equations can be used to describe the non-acoustic component of the motion. To this end we identify them with the Favre filtered Navier Stokes equations by interpreting the over bars in equations [2.2]-[2.4] to be the filtered variables

$$\overline{f}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_V g(\mathbf{x} - \boldsymbol{\xi}, t - \tau) f(\boldsymbol{\xi}, \tau) d\boldsymbol{\xi} d\tau, \quad [3.1]$$

and the tildes to be the Favre filtered variables

$$\tilde{\bullet} \equiv (\overline{\rho \bullet}) / \bar{\rho} \quad [3.2]$$

for all variables except \tilde{h}_o , which is defined by

$$\tilde{h}_o \equiv \tilde{h} + \frac{1}{2}\tilde{v}^2 \quad [3.3]$$

where f denotes any combination of the variables ρ , v_i , p , and h , the dot is a place holder for these quantities and the kernel $g(\mathbf{x}, t)$ of generalized filter

space- time filter (Aldama,1990) can be any generalized function (in the distribution sense) of space and time. We *do not*, however, required to satisfy the usual normalization condition

$$\int_{-\infty}^{\infty} \int_V g(\mathbf{x}, t) d\mathbf{x} dt = 1 \quad [3.4]$$

The base flow source strength \tilde{e}_{vi} is then given by (Goldstein, 2003)

$$\tilde{e}_{ij} = -\bar{\rho}(\widetilde{v_i v_j} - \tilde{v}_i \tilde{v}_j) + \frac{\gamma-1}{2} \delta_{ij} \bar{\rho}(\widetilde{v^2} - \tilde{v}^2) \quad [3.5]$$

and

$$\tilde{e}_{4i} = -(\gamma-1)\bar{\rho}[\widetilde{h_0 v_i} - \tilde{h}_0 \tilde{v}_i - (\widetilde{v_i v_j} - \tilde{v}_i \tilde{v}_j) \tilde{v}_j] + \frac{\gamma-1}{2} \bar{\rho}(\widetilde{v^2} - \tilde{v}^2) \tilde{v}_i \quad [3.6]$$

4. DECOMPOSITION OF THE FLOW INTO RADIATING AND NON-RADIATING COMPONENTS

In order to actually construct the non-radiating base flow, we take the Fourier transform of [3.1] and use the convolution theorem to show that

$$\bar{F}(\mathbf{k}, \omega) = (2\pi)^4 G(\mathbf{k}, \omega) F(\mathbf{k}, \omega), \quad [4.1]$$

where the capital letters denote the space time Fourier transforms

$$F(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int_V \int_{-\infty}^{\infty} e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega t)} f(\mathbf{x}, t) d\mathbf{x} dt \quad [4.2]$$

of the corresponding lower case symbols.

It follows that it is only necessary to select a filter kernel $g(\mathbf{x}, t)$ that has no wavenumber components lying on the spherical surface of radius ω/c_0 in order to insure that the Fourier transform $\bar{F}(\mathbf{k}, \omega)$ of the corresponding filtered variable $\bar{f}(\mathbf{x}, t)$ has a similar property. Then, as noted in the introduction, using such a filter to determine the base flow equations will cause that flow to be non-radiating. Solutions of model problems suggest that the Fourier transforms $F(\mathbf{k}, \omega)$ of the original Navier -Stokes variables (ρ, v_i, p , and h) may be singular

on this surface, which could place some restrictions on the form of the filter, but would not invalidate the result.

Since the radiating sphere in \mathbf{k} - space corresponds to the 45° lines $k_\pm = 0$, where

$$k_\pm \equiv k \pm (\omega / c_0) \quad [4.3]$$

in the ω / c_0 , $k = |\mathbf{k}|$ -space shown in figure 2, this is equivalent to requiring that $G(\mathbf{k}, \omega)$ vanish on those lines. An appropriate filter g can be constructed by putting

$$(2\pi)^4 G(\mathbf{k}, \omega) = [1 - \Theta(\Delta k_+)] [1 - \Theta(\Delta k_-)] \quad [4.4]$$

where $\Delta \geq 0$ is a parameter

$$\Theta(\kappa \Delta) \equiv \frac{\sin(\kappa \Delta)}{\kappa \Delta} = \frac{1}{2\Delta} \int_{-\infty}^{\infty} e^{i\kappa t} [H(t + \Delta) - H(t - \Delta)] dt \quad [4.5]$$

and

$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad [4.6]$$

is the Heaviside unit function. Taking inverse transforms shows

$$\theta(t, \Delta) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa t} \Theta(\kappa \Delta) d\kappa = \frac{H(t + \Delta) - H(t - \Delta)}{2\Delta} \quad [4.7]$$

Figure 3 is a plot of $\Theta(\kappa \Delta)$ vs. κ for various values of Δ . Notice that it becomes more and more concentrated around $\kappa = 0$ when Δ becomes large. The $\Theta(k_\pm \Delta)$, therefore, become concentrated around the 45° lines in the $k - \omega$ plane when $\Delta \rightarrow \infty$, which means that the base flow will contain all of the non-radiating components of the motion in this limit.

It now follows from equation [4.7] and APPENDIX A that the filter g is given by

$$g(\mathbf{x}, t) = \delta(t) \delta(\mathbf{x}) + \frac{c_0}{2\pi x} \frac{\partial}{\partial x} \left[\theta\left(\frac{x_-}{2}, \Delta\right) \delta(x_+) + \theta\left(\frac{x_+}{2}, \Delta\right) \delta(x_-) - \frac{1}{2} \theta\left(\frac{x_-}{2}, \Delta\right) \theta\left(\frac{x_+}{2}, \Delta\right) \right] \quad [4.8]$$

where x_{\pm} denote the characteristic coordinates

$$x_{\pm} \equiv x \pm c_0 t \quad [4.9]$$

And since equation [3.1] can also be written as

$$\bar{f}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_V g(\xi, \tau) f(\mathbf{x} - \xi, t - \tau) d\xi d\tau, \quad [4.10]$$

it follows that the filtered variables $\bar{f}(\mathbf{x}, t)$ are given by

$$\begin{aligned} \bar{f}(\mathbf{x}, t) = f(\mathbf{x}, t) + \\ \frac{c_0}{2\pi} \int_{-\infty}^{\infty} \int_V \frac{1}{\xi} \frac{\partial}{\partial \xi} \left[\theta\left(\frac{\xi_-}{2}, \Delta\right) \delta(x_+) + \theta\left(\frac{\xi_+}{2}, \Delta\right) \delta(x_-) - \frac{1}{2} \theta\left(\frac{\xi_-}{2}, \Delta\right) \theta\left(\frac{\xi_+}{2}, \Delta\right) \right] f(\mathbf{x} - \xi, t - \tau) d\xi d\tau \end{aligned} \quad [4.11]$$

where ξ_{\pm} denote the dummy integration variables

$$\xi_{\pm} \equiv \xi \pm c_0 \tau \quad [4.12]$$

corresponding to the characteristic variables [4.9]. Integrating this result by parts shows that

$$\bar{f}(\mathbf{x}, t) = f(\mathbf{x}, t) +$$

$$\begin{aligned}
& -\frac{1}{2\pi} \left\{ \frac{1}{2\Delta} \int_0^\Delta \int_\Omega \frac{\partial}{\partial \xi} \left\{ \xi \left[f(\mathbf{x} - \boldsymbol{\xi}, t - \tau) + f(\mathbf{x} - \boldsymbol{\xi}, t + \tau) \right] \right\} \right\} \Big|_{\tau=\xi/c_0} d\xi d\Omega \\
& -\frac{c_0}{2} \int_{-\infty}^{\infty} \int_0^\infty \int_\Omega \theta\left(\frac{\xi_-}{2}, \Delta\right) \theta\left(\frac{\xi_+}{2}, \Delta\right) \frac{\partial}{\partial \xi} \xi f(\mathbf{x} - \boldsymbol{\xi}, t - \tau) d\tau d\xi d\Omega \Big\} \quad [4.13]
\end{aligned}$$

where the Ω - integration is over the unit sphere in ξ -space. This result will account for all of the non-radiating components of $f(\mathbf{x}, t)$ when $\Delta \rightarrow \infty$.

Notice that

$$\begin{aligned}
& \frac{\partial}{\partial \xi} \left\{ \xi \left[f(\mathbf{x} - \boldsymbol{\xi}, t - \tau) + f(\mathbf{x} - \boldsymbol{\xi}, t + \tau) \right] \right\} \Big|_{\tau=\xi/c_0} \\
& \neq \frac{\partial}{\partial \xi} \left\{ \xi \left[f(\mathbf{x} - \boldsymbol{\xi}, t - \xi/c_0) + f(\mathbf{x} - \boldsymbol{\xi}, t + \xi/c_0) \right] \right\} \quad [4.14]
\end{aligned}$$

so that the next to last integration in [4.13] cannot, in general, be carried out in closed form. This can, however, always be done when $|\mathbf{x}|$ is in the far field, and APPENDIX B uses Lighthill's equation to show that the filter [4.8] eliminates the entire acoustic field in this case.

The residual variables $f'(\mathbf{x}, t) \equiv f(\mathbf{x}, t) - \bar{f}(\mathbf{x}, t)$ now correspond to the negative of the curly bracket term in [4.13] and will, therefore, account for the entire radiating component of the motion when $\Delta \rightarrow \infty$. But their space-time Fourier transforms will then occupy zero volume in wavenumber-frequency space, which should not be a problem, since the most widely used filter, namely the time average

$$\bar{\bullet}(\mathbf{x}) \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \bullet(\mathbf{x}, t) dt, \quad [4.15]$$

also has this property (because it excludes all frequencies that do not lie on the $\omega = 0$ axis). The residual variables are still determined by [2.6] to [2.8], whose right hand sides can now be interpreted as *pure* acoustic sources (in the sense that they generate only the acoustic component of the motion). This interpretation would not, of course, be valid if Δ were finite, because the residual variables would then have to account for some of the non-radiating components of the motion. However, the acoustic and non-acoustic components of the motion will still be fully coupled even when the non-acoustic component is eliminated from the residual variables. They can, of course, be decoupled by modeling the base

flow source terms [3.5] and [3.6], which would greatly simplify the computation of that flow. This (approximate) partial decoupling is similar to the (exact) partial decoupling found by Goldstein (1978) for the small amplitude vortical and entropic motion on a potential flow—but certainly not the complete decoupling found by Kovasznay (1953).

The complete residual equation source strength e''_{vi} involves both base flow and quadratically non-linear residual components. The latter, which can either represent true sound sources or non-linear propagation effects, are likely to be small at subsonic and moderately supersonic speeds, since, as noted above, only a very small fraction of the flow energy can radiate. But this would imply that the base flow source strengths [3.5] and [3.6] are responsible for generating both the radiating and non-components of the motion. (Notice that the first terms in these equations are non-radiating but the second terms, which involve quadratic interactions between the non-radiating components, can generate radiating wave numbers.) This would certainly make it very difficult to model these quantities—especially if the models could only depend on the base flow variables. But this would decouple the base flow solution from the residual (radiating) motion and the residual equations would then be an inhomogeneous linear system with known source terms (that are determined by the base flow solution). The right hand sides of these equations could then be interpreted as pure acoustic sources in the strict classical acoustics sense. *But the interpretation would be based on an approximate result!* The main difficulty is that the modeled stresses generate both the radiating and non-radiating components of the motion—which is a good indication that this approach may be too subtle to implement numerically. It is, of course, possible to move the residual stresses to the left side of the equations and calculate the sound from the full nonlinear equations, but this would make the present approach more complicated and computationally more expensive than solving the original Navier–Stokes equations.

5. MORE COMPUTATIONALLY VIABLE APPROACHES

While the present result is certainly of theoretical interest, it can probably not be implemented without first introducing an appropriate source model to close the base flow equations. But since that has not, as yet, been accomplished, the only viable options for creating non-radiating base flows is to make them behave incompressibly (Hardin & Pope, 1994; Fedorchenko, 2001, Goldstein, 2003) or to be less ambitious in the choice for the filter. We focus here in the latter because the former is probably inappropriate for the high Mach numbers of technological interest. But the only non-radiating (compressible) flows with well developed source models are the steady (time average) flows corresponding to the pure time average filter [4.15]. The base flow equations would then be the usual Reynolds averaged Navier Stokes (RANS) equations (Pope, 2000) and the residual source strength [2.13] - [2.15], [3.5] and [3.6] would be given by (Goldstein, 2003)

$$e''_{\lambda j} = e'_{\lambda j} - \overline{e'_{\lambda j}}, \quad [5.1]$$

which shows that it now has zero time average as would be expected from a true acoustic source.

Unfortunately, the residual motion is now the entire unsteady flow (which certainly has significant non-acoustic components) and the dominant contribution to the residual sources [2.14] and [2.15] comes from the residual stresses, which can no longer be determined from the base flow. It is, therefore, necessary to close the residual equations by introducing an appropriate model for these stresses, which may, however, depend on the base flow solution. This is a major disadvantage of this approach, but it can be shown (Goldstein and Leib, 2007) by using the vector Green's function $g_{\lambda v}(\mathbf{x}, t | \mathbf{y}, \tau)$ to obtain an expression for the pressure-like variable p'_e in terms of the residual source strength [5.1], that the far

field pressure autocovariance $\overline{p'^2}(\mathbf{x}, t) \equiv (2T)^{-1} \int_{-T}^T p'(\mathbf{x}, t) p'(\mathbf{x}, t + t_0) dt$, which is the quantity of principal interest in jet noise calculations, is given by the convolution product

$$\overline{p'^2}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_V \overline{\gamma}_{vj\mu l}(\mathbf{x} | \mathbf{y}; \boldsymbol{\eta}, t + \tau) \mathcal{R}_{vj\mu l}(\mathbf{y}; \boldsymbol{\eta}, \tau) d\mathbf{y} d\boldsymbol{\eta} d\tau \quad [5.2]$$

of the propagator

$$\overline{\gamma}_{vj\mu l}(\mathbf{x} | \mathbf{y}; \boldsymbol{\eta}, t + \tau) \equiv \int_{-\infty}^{\infty} \gamma_{vj}(\mathbf{x} | \mathbf{y}, t_1 + t + \tau) \gamma_{\mu l}(\mathbf{x} | \mathbf{y} + \boldsymbol{\eta}, t_1) dt_1 \quad [5.3]$$

where

$$\gamma_{\mu j}(\mathbf{x}, t | \mathbf{y}, \tau) \equiv \frac{\partial g_{4\mu}(\mathbf{x}, t | \mathbf{y}, \tau)}{\partial y_j} - (\gamma - 1) \frac{\partial \tilde{v}_\mu}{\partial y_j} g_{44}(\mathbf{x}, t | \mathbf{y}, \tau), \quad \tilde{v}_4 \equiv 0 \quad [5.4]$$

depends only on the mean flow, with a modified Reynolds stress correlation tensor $\mathcal{R}_{vj\mu l}(\mathbf{y}; \boldsymbol{\eta}, \tau)$, which (in the absence of viscosity) is, in turn, related to the generalized velocity/enthalpy Reynolds stress autocovariance tensor

$$R_{vj\mu l}(\mathbf{y}; \boldsymbol{\eta}, \tau) \equiv \frac{1}{2T} \int_{-T}^T [\rho v'_v v'_j - \overline{\rho v'_v v'_j}](\mathbf{y}, \tau_0) [\rho v'_\mu v'_l - \overline{\rho v'_\mu v'_l}](\mathbf{y} + \boldsymbol{\eta}, \tau_0 + \tau) d\tau_0 \quad [5.5]$$

which is about as close as you can get to what is actually measured in a turbulent flow, by the simple linear transform

$$\mathcal{R}_{vj\mu l} = R_{vj\mu l} - \frac{\gamma-1}{2} \left(\delta_{vj} R_{kk\mu l} + \delta_{\mu l} R_{vjkk} \right) + \left(\frac{\gamma-1}{2} \right)^2 \delta_{vj} \delta_{\mu l} R_{iikk} \quad [5.6]$$

where $\delta_{\mu l}$ denotes the four dimensional Kronecker delta. This shows, among other

things, that $\overline{p^2}(\mathbf{x}, t)$ depends only on the two point correlations of the turbulent stresses and not on their instantaneous values, which greatly simplifies the modeling requirements. But, as noted above, the models must represent the entire turbulent flow and, therefore, cannot be expected to be very universal.

This latter difficulty can, in principle, be overcome by choosing the filter [3.1] to be purely spatial, which would turn the base flow equations into the usual large eddy simulation (LES) equations. These equations are usually closed by introducing appropriate models for the base flow stresses, which would again produce a partial decoupling of the base flow and residual motions. But the original objective of dividing the motion into acoustic and non-acoustic components is now completely lost, since the base flow is now radiating.

The residual equations can again be closed by introducing an appropriate model for the residual stresses (which may again depend on the base flow solution) and the result can be used to calculate the sound from the unresolved scales (Bodony and Lele, 2002). But the far field pressure autocovariance $\overline{p^2}(\mathbf{x}, t)$ will depend on the instantaneous values of these stresses, which are much more difficult to model than the lower order turbulence statistics embodied in the Reynolds stress autocovariance tensor. And since this will be the case whenever the base flow is unsteady, it may be that the steady base flow approach, which represents the current state of the art (Khavaran and Bridges, 2004; Goldstein, and Leib, 2007), is actually the best methodology to use at this point in time—especially since the basic formula [5.2] provides an effectively exact relation between the type of quantities that are actually measured in real turbulent flows.

But the question then arises as to whether this, necessarily empirical, approach can be used to identify the sound sources and, more importantly, sound generation mechanisms. The answer is that it may (depending on how well the source terms can be modeled) be able to provide some useful insights into the nature of these sources and/or mechanisms, but is inherently incapable of providing exact results—primarily because it does not completely eliminate all of the source identification issues associated with the Lighthill approach. This also has important experimental implications because it is only the Reynolds stresses themselves and not the radiating component of these stresses that can actually be measured in a turbulent flow.

6. CONCLUSIONS

Lighthill (1952) argued that the strength of his quadrupole source could be obtained to a good approximation by calculating its value for an equivalent flow devoid of sound. The present result provides an analytical basis for that idea. Since the Fourier transform filter widths can, be made arbitrarily small, our analysis shows that the base flow can, in principle, be chosen to be the entire non-radiating component of the motion with the residual flow containing only radiating components. And since only a small fraction of the flow energy gets radiated as sound, the latter should be small compared to the base flow component and should therefore be almost completely generated by the base flow source term, which means it will be determined by a known source term in a nearly linear set of equations if the base flow equations could be closed by introducing an appropriate source model for that flow. The acoustic sources could then be unambiguously identified in the classical acoustics sense. But no base flow closure model has as ever been proposed!

APPENDIX A

We can simplify the results by allowing k to become negative. Then for any filter $G(\mathbf{k}, \omega) = G(k, \omega)$ such that $G(-k, \omega) = G(k, \omega)$ (where $G(k, \omega)$ has the obvious meaning)

$$\begin{aligned}
 g(\mathbf{x}, t) &= \int_V \int_{-\infty}^{\infty} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} G(k, \omega) d\mathbf{k} d\omega = \int_V \int_{-\infty}^{\infty} e^{i(kx \cos \theta - \omega t)} G(k, \omega) \sin \theta d\theta d\psi k^2 dk d\omega \\
 &= \frac{2\pi}{ix} \int_{-\infty}^{\infty} \int_0^{\infty} \left[e^{i(kx - \omega t)} - e^{-i(kx + \omega t)} \right] G(k, \omega) k dk d\omega = \frac{2\pi}{ix} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(kx - \omega t)} G(k, \omega) k dk d\omega \\
 &= \frac{\pi c_0}{2ix} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i}{2} [k_+(x - c_0 t) + k_-(x + c_0 t)]} G(k, \omega) (k_+ + k_-) dk_+ dk_- \quad [\text{A.1}]
 \end{aligned}$$

where, as noted above, we have simplified the notation by allowing k to take on negative values, chosen the polar axis in the θ, ψ, k spherical coordinate system

to be in the x -direction and introduced the new integration variables k_{\pm} given by [4.3]. But it follows from equation [4.7] that

$$\begin{aligned}
& \frac{\pi c_0}{2ix} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i}{2}[k_+(x-c_0t)+k_-(x+c_0t)]} \Theta(k_{\pm}\Delta)(k_+ + k_-) dk_+ dk_- = \\
& \frac{\pi c_0}{2ix} \left[\int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\pm}(x \mp c_0t)} \Theta(k_{\pm}\Delta) k_{\pm} dk_{\pm} \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\mp}(x \pm c_0t)} dk_{\mp} + \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\pm}(x \mp c_0t)} \Theta(k_{\pm}\Delta) dk_{\pm} \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\mp}(x \pm c_0t)} k_{\mp} dk_{\mp} \right] \\
& = \frac{\pi c_0}{2ix} \left[\frac{1}{2i\Delta} \int_{-\infty}^{\infty} \left\{ e^{ik_{\pm}[(x \mp c_0t)/2 + \Delta]} - e^{ik_{\pm}[(x \mp c_0t)/2 - \Delta]} \right\} dk_{\pm} \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\mp}(x \pm c_0t)} dk_{\mp} \right. \\
& \quad \left. + \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\pm}(x \mp c_0t)} \Theta(k_{\pm}\Delta) dk_{\pm} \int_{-\infty}^{\infty} e^{\frac{i}{2}k_{\mp}(x \pm c_0t)} k_{\mp} dk_{\mp} \right] \\
& = \frac{\pi c_0}{2ix} \left\{ \frac{(2\pi)^2}{i\Delta} \left[\delta\left(\frac{x \mp c_0t}{2} + \Delta\right) - \delta\left(\frac{x \mp c_0t}{2} - \Delta\right) \right] \delta(x \pm c_0t) \right. \\
& \quad \left. + 2 \frac{(2\pi)^2}{i\Delta} \left[H\left(\frac{x \mp c_0t}{2} + \Delta\right) - H\left(\frac{x \mp c_0t}{2} - \Delta\right) \right] \frac{d}{dx} \delta(x \pm c_0t) \right\} \\
& = -\frac{c_0(2\pi)^3}{2\Delta x} \frac{d}{dx} \left\{ \left[H\left(\frac{x \mp c_0t}{2} + \Delta\right) - H\left(\frac{x \mp c_0t}{2} - \Delta\right) \right] \delta(x \pm c_0t) \right\} \quad [\text{A.2}]
\end{aligned}$$

and that

$$\frac{\pi c_0}{2ix} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i}{2}[k_+(x-c_0t)+k_-(x+c_0t)]} \Theta(k_+\Delta) \Theta(k_-\Delta) (k_+ + k_-) dk_+ dk_- =$$

$$-\frac{(2\pi)^3 c_0}{2x} \frac{\partial}{\partial x} \theta\left(\frac{x-c_0 t}{2}, \Delta\right) \theta\left(\frac{x-c_0 t}{2}, \Delta\right) \quad [\text{A.3}]$$

where $\delta(x)$ denotes the Delta function in the usual notation.

APPENDIX B

Lighthill's equation

$$\frac{\partial^2 \rho}{\partial t^2} - c_o^2 \frac{\partial^2 \rho}{\partial x_j \partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} \quad [\text{B.1}]$$

where

$$T_{ij} \equiv \rho v_i v_j + \delta_{ij} (p - c_o^2 \rho) \quad [\text{B.2}]$$

is the Lighthill stress tensor with the viscous terms (which are believed to play an insignificant role in the sound generation process) omitted can be used to directly verify that the filter [4.8] eliminates the radiating component of the motion.

This equation (which is an exact result) can be solved to obtain (recall that we are assuming the flow to be unbounded)

$$\rho(\mathbf{x}, t) = \frac{1}{4\pi c_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}(\mathbf{y}, t - |\mathbf{x} - \mathbf{y}|/c_o)}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \quad [\text{B.3}]$$

which behaves like

$$c_o^2 \rho(\mathbf{x}, t) \rightarrow \frac{x_i x_j}{4\pi c_o^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int_V T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) d\mathbf{y} \quad \text{as } |\mathbf{x}| \rightarrow \infty \quad [\text{B.4}]$$

and it follows from classical acoustics that the remaining dependent variables will exhibit similar behavior. Equation [B.4] implies that

$$c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t \mp \tau) \rightarrow \frac{x_i x_j}{4\pi c_o^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int_V T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \boldsymbol{\xi}}{|\mathbf{x}| c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) d\mathbf{y} \quad \text{as } |\mathbf{x}| \rightarrow \infty \quad [\text{B.5}]$$

and therefore that

$$\begin{aligned}
& -\frac{1}{4\pi\Delta} \int_0^\Delta \int_\Omega \frac{\partial}{\partial \xi} \left\{ \xi \left[c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t - \tau) + c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t + \tau) \right] \right\} \Big|_{\tau=\xi/c_o} d\xi d\Omega \rightarrow \\
& -\frac{x_i x_j}{4\pi c_o^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \frac{1}{4\pi\Delta} \int_V \int_0^\Delta \int_\Omega \frac{\partial}{\partial \xi} \left\{ \xi \left[T_{ij} \left(\mathbf{y}, t - \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \boldsymbol{\xi}}{|\mathbf{x}|c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) \right. \right. \\
& \quad \left. \left. + T_{ij} \left(\mathbf{y}, t + \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \boldsymbol{\xi}}{|\mathbf{x}|c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) \right] \right\} \Big|_{\tau=\xi/c_o} d\xi d\Omega d\mathbf{y} \\
& \quad \quad \quad as |\mathbf{x}| \rightarrow \infty \quad [B.6]
\end{aligned}$$

But since introducing the new dependent variable $\eta \equiv \xi \cos \theta$ shows that

$$\begin{aligned}
& \int_\Omega \frac{\partial}{\partial \xi} \xi T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \boldsymbol{\xi}}{|\mathbf{x}|c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) d\Omega = \\
& 2\pi \int_0^\pi \frac{\partial}{\partial \xi} \xi T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\xi \cos \theta}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) \sin \theta d\theta \\
& = -\frac{2\pi}{\xi} \int_{-\xi}^{\xi} \frac{\partial}{\partial \eta} \eta T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\eta}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) d\eta = \\
& 2\pi \left[T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} - \frac{\xi}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) + T_{ij} \left(\mathbf{y}, t \mp \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\xi}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|c_o} \right) \right] \quad [B.7]
\end{aligned}$$

it now follows that

$$\begin{aligned}
& -\frac{1}{4\pi\Delta} \int_0^\Delta \int_\Omega \frac{\partial}{\partial \xi} \left\{ \xi \left[c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t - \tau) + c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t + \tau) \right] \right\} \Big|_{\tau=\xi/c_o} d\xi d\Omega \rightarrow \\
& -\frac{x_i x_j}{4\pi c_o^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \frac{1}{\Delta} \int_V \int_0^\Delta \left[T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) + \frac{1}{2} T_{ij} \left(\mathbf{y}, t - 2\frac{\xi}{c_o} - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) \right. \\
& \quad \left. + \frac{1}{2} T_{ij} \left(\mathbf{y}, t + 2\frac{\xi}{c_o} - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) \right] d\xi d\mathbf{y} \quad [\text{B.8}]
\end{aligned}$$

and that

$$\begin{aligned}
& -\frac{c_0}{4\Delta} \int_0^\infty \int_{-\infty}^\infty \int_\Omega \left[H \left(\frac{\xi - c_0 \tau}{2} + \Delta \right) - H \left(\frac{\xi - c_0 \tau}{2} - \Delta \right) \right] \\
& \quad \times \left[H \left(\frac{\xi + c_0 \tau}{2} + \Delta \right) - H \left(\frac{\xi + c_0 \tau}{2} - \Delta \right) \right] \frac{\partial}{\partial \xi} \xi T_{ij} \left(\mathbf{y}, t - \tau - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \boldsymbol{\xi}}{|\mathbf{x}| c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) d\tau d\xi d\Omega \\
& = -\frac{\pi}{8\Delta} \int_{-2\Delta}^{2\Delta} \int_{-2\Delta}^{2\Delta} \left[T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} - \frac{\xi_+}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) + T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} + \frac{\xi_-}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) \right] d\xi_+ d\xi_- \\
& = -\pi \int_{-\Delta}^{\Delta} \left[T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} - \frac{2\xi}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) + T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} + \frac{2\xi}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) \right] d\xi \quad [\text{B.9}] \\
& = -2\pi \int_{-\Delta}^{\Delta} T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} - \frac{2\xi}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) d\xi
\end{aligned}$$

where we have again simplified the result by using symmetry to artificially extend the integration range to negative ξ -values. It now follows that

$$\begin{aligned}
& -\frac{1}{4\pi\Delta} \left\{ \int_0^\Delta \int_\Omega \frac{\partial}{\partial \xi} \left\{ \xi \left[c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t - \tau) + c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t + \tau) \right] \right\} \right\} \Big|_{\tau=\xi/c_o} d\xi d\Omega \\
& -\frac{1}{4\Delta} \int_{-\infty}^{\infty} \int_0^\infty \int_\Omega \left[H\left(\frac{\xi - c_o \tau}{2} + \Delta\right) - H\left(\frac{\xi - c_o \tau}{2} - \Delta\right) \right] \\
& \quad \times \left[H\left(\frac{\xi + c_o \tau}{2} + \Delta\right) - H\left(\frac{\xi + c_o \tau}{2} - \Delta\right) \right] \frac{\partial}{\partial \xi} \xi c_o^2 \rho(\mathbf{x} - \boldsymbol{\xi}, t - \tau) d\tau d\xi d\Omega \Big\} \rightarrow \\
& -\frac{x_i x_j}{4\pi c_o^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int_V T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_o} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_o} \right) d\mathbf{y} = -c_o^2 \rho(\mathbf{x}, t) \tag{B.10}
\end{aligned}$$

Then since the remaining dependent variables must also behave like[B.4], this shows that the filter [4.8] eliminates the acoustic field.

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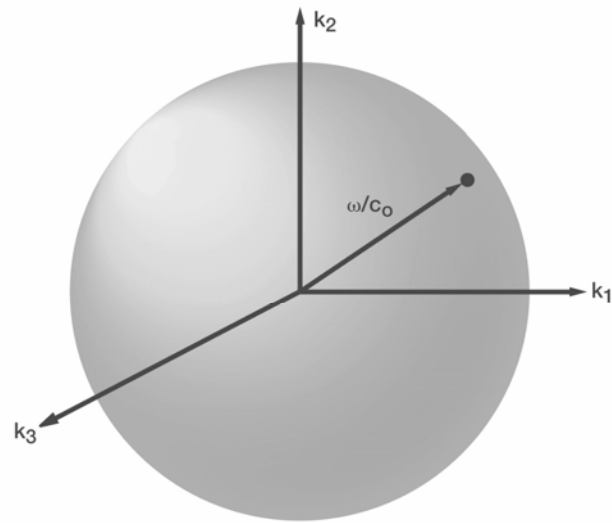


Figure 1.—Surface of sound producing wave numbers.

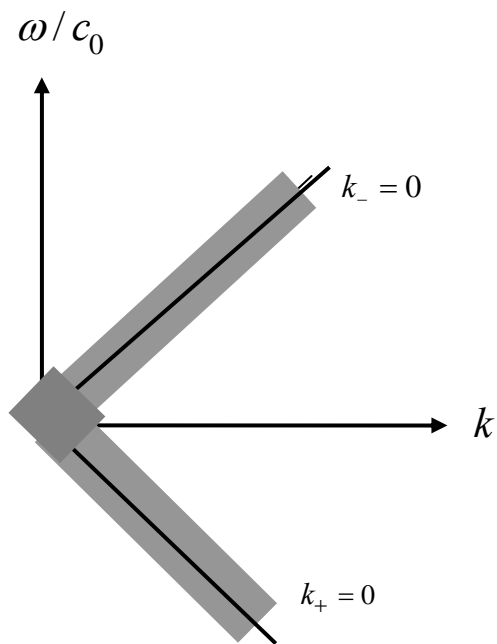


Figure 2.--Radiating elements in $\omega/c_0 - k$ space

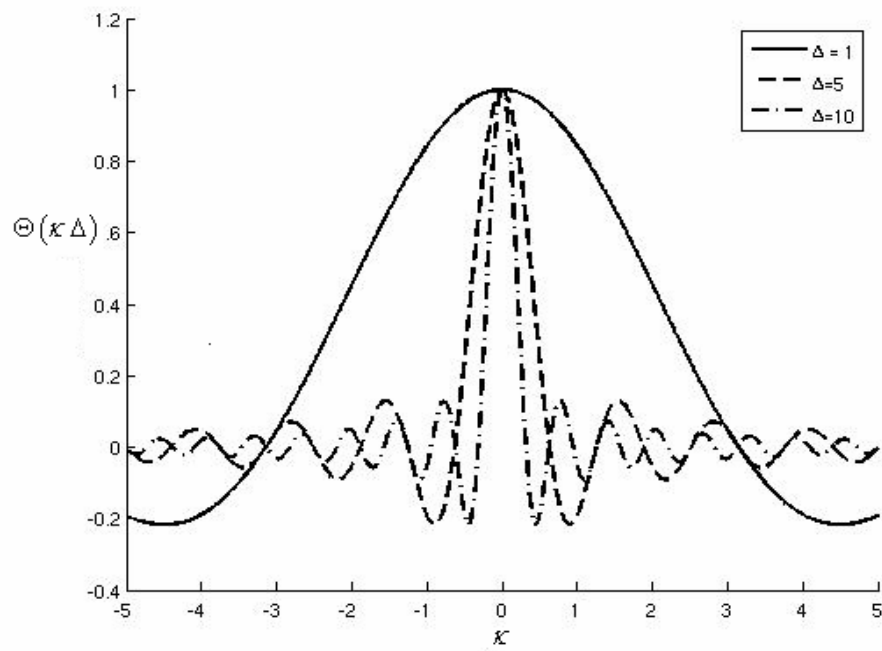


Figure 3.--Plot of $\Theta(\kappa \Delta)$ vs. κ for various values of Δ .